Loop integrals, integration-by-parts and MATAD

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Outline

I. Integration-by-parts

II. MATAD

III. Examples

I.

Integration-by-Parts (IBP)

[Many thanks to Robert Harlander for the possibility to recycle some of his transparencies.]

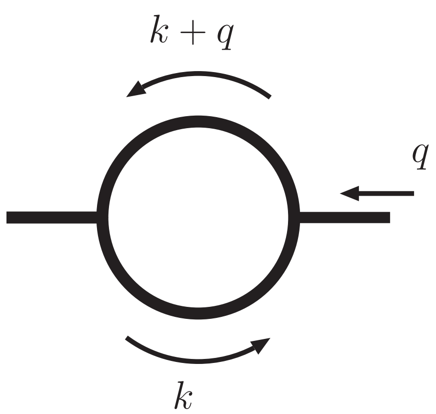
A. Notation

- $$F(n_1,\ldots,n_N) = \int \frac{\mathrm{d}^d k_1 \cdots \mathrm{d}^d k_l}{(-p_1^2 + m_1^2)^{n_1} \cdots (-p_N^2 + m_N^2)^{n_N}}$$

N	—	number of propagators
n_i	—	indices
l	—	number of loops
p_i	—	linear combinations of loop momenta k_i and external momenta q_i

- Dimensional Regularization
  - $d = 4 - 2\epsilon$
  - scaleless integrals $= 0$

B. Example: 1-loop self energy



$$= F(n_1, n_2) = \int \mathrm{d}^d k \frac{1}{(-p_1^2 + m_1^2)^{n_1} (-p_2^2 + m_2^2)^{n_2}}$$

$$p_1 = k\,, \qquad p_2 = k + q$$

special case:

- $m_1 = m_2 = 0$: $\quad [n_1 = a, n_2 = b]$

$$I_q(a,b) = i\pi^{d/2} (-q^2)^{d/2-a-b} \cdot \hat{I}(a,b)$$

$$\hat{I}(a,b) = \frac{\Gamma(a+b-d/2)\,\Gamma(d/2-a)\,\Gamma(d/2-b)}{\Gamma(a)\,\Gamma(b)\,\Gamma(d-a-b)}$$

Example: 1-loop self energy

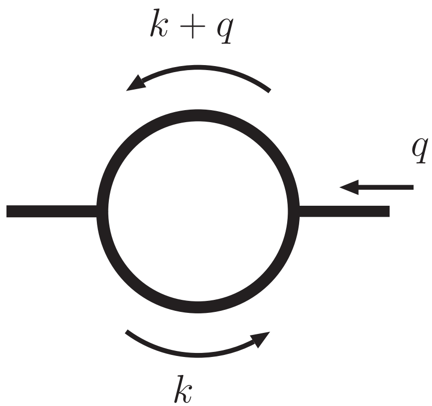
$$\hat{I}(a,b) = \frac{\Gamma(a+b-d/2)\,\Gamma(d/2-a)\,\Gamma(d/2-b)}{\Gamma(a)\,\Gamma(b)\,\Gamma(d-a-b)}$$

- remarks:
  - valid for (almost) all a, b, d
  - expansion in ϵ through $z\Gamma(z) = \Gamma(1+z)$ and

$$\Gamma(1+\epsilon) = 1 - \epsilon\gamma_E + \frac{\epsilon^2}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right) + \ldots$$

(γ_E is cancelled by our normalization)

Example: 1-loop self energy

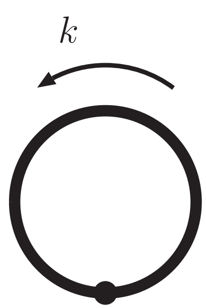


$$= F(n_1, n_2) = \int \mathrm{d}^d k \frac{1}{(-p_1^2 + m_1^2)^{n_1}(-p_2^2 + m_2^2)^{n_2}}$$

$$p_1 = k\,, \qquad p_2 = k + q$$

special case:

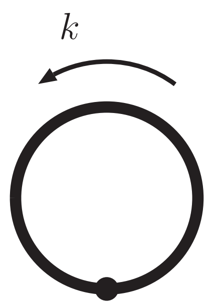
- $n_2 = 0$ [equivalently: $q = 0$, $m_1 = m_2$]



$$= F(n) = \int \mathrm{d}^d k \frac{1}{(-k^2 + m^2)^n}$$

$$= i\pi^{d/2}(m^2)^{d/2-n}\frac{\Gamma(n-d/2)}{\Gamma(n)}$$

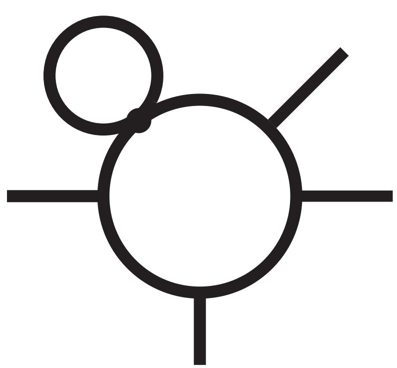
Example: 1-loop self energy



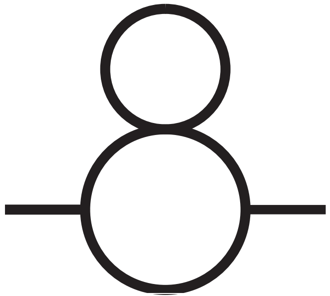
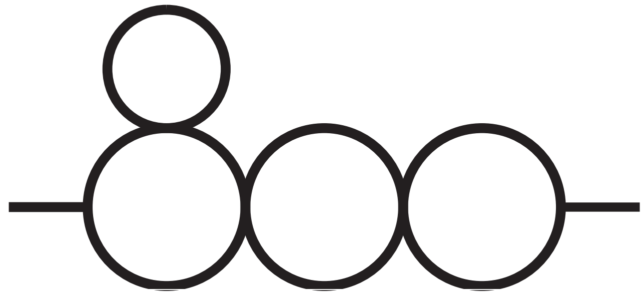
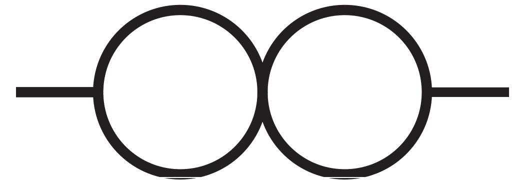
$$= F(n) = \int \mathrm{d}^d k \, \frac{1}{(-k^2 + m^2)^n}$$

$$= i\pi^{d/2} (m^2)^{d/2-n} \frac{\Gamma(n-d/2)}{\Gamma(n)}$$

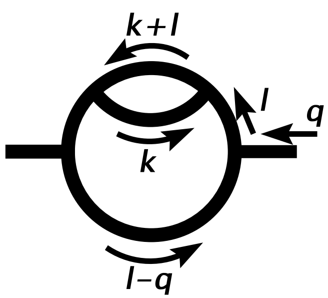
- note: massless tadpoles $= 0$ (scaleless integrals)

e.g.  $= 0$ in massless case

C. Products and convolutions of 1-loop

-   

$\rightarrow$ factorization

- 

$$= \int \mathrm{d}^d l \frac{1}{(l^2)^{n_1+n_3}[(l-q)^2]^{n_4}} \underbrace{\int \mathrm{d}^d k \frac{1}{(k^2)^{n_5}[(k+l)^2]^{n_2}}}_{\sim (l^2)^{d/2-n_5-n_2}\hat{I}(n_2,n_5)}$$

$$\Bigg|_{a=n_1+n_3+n_2+n_5-d/2} \quad \sim \hat{I}(n_2,n_5) \int \mathrm{d}^d l \frac{1}{(l^2)^a[(l-q)^2]^{n_4}}$$

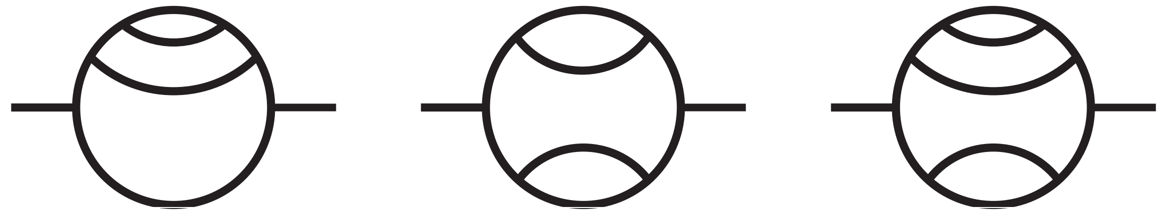
$$\sim (q^2)^{d-\sum n_i} \cdot \hat{I}(n_1+n_3+n_2+n_5-d/2, n_4) \cdot \hat{I}(n_2,n_5)$$

[Euclidian momenta]

Products and convolutions of 1-loop (2)

- analogously:

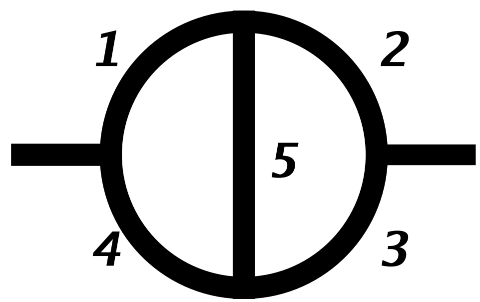
  (for $m = 0$)

  convolutions of 1-loop self-energy diagrams

- however:

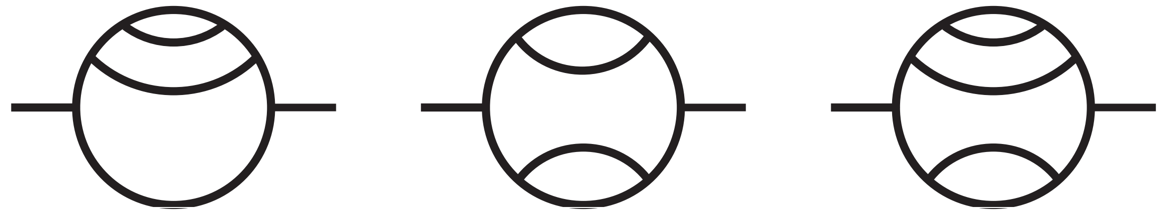
  genuine 2-loop:

  "T_1"–topology

Products and convolutions of 1-loop (2)

- analogously:
  (for $m = 0$)



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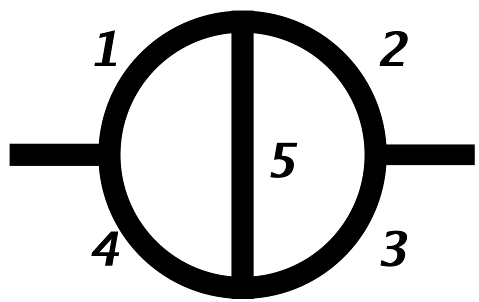
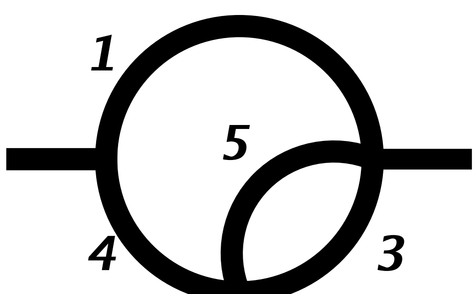


"T_1"–topology

but: shrinking any line $\Rightarrow$ convolution of 1-loop!



D. IBP relations

Dimensional regularization:

- Integrals are finite
- Surface terms vanish

➪ $$\boxed{\int \mathrm{d}^d k \frac{\partial}{\partial k_\mu} p_\mu f(k,p,\ldots) = 0}$$

p_μ: loop or external momentum

Example: $F(n) = \int \mathrm{d}^d k \frac{1}{(-k^2+m^2)^n}$

Consider:

$$0 = \int \mathrm{d}^d k \frac{\partial}{\partial k_\mu} k_\mu \frac{1}{(-k^2+m^2)^n}$$

D. IBP relations

Example: $F(n) = \int \mathrm{d}^d k \frac{1}{(-k^2+m^2)^n}$

$$
\begin{aligned}
0 &= \int \mathrm{d}^d k \frac{\partial}{\partial k_\mu} k_\mu \frac{1}{(-k^2+m^2)^n} \\
&= \int \mathrm{d}^d k \frac{d}{(-k^2+m^2)^n} + \frac{k_\mu(-n)(-2k^\mu)}{(-k^2+m^2)^{(n+1)}} \\
&= \\
&=
\end{aligned}
$$

D. IBP relations

Example: $F(n) = \int \mathrm{d}^d k \frac{1}{(-k^2+m^2)^n}$

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&= \int \mathrm{d}^d k \frac{d-2n}{(-k^2+m^2)^n} + \frac{2nm^2}{(-k^2+m^2)^{(n+1)}} \\
&=
\end{aligned}
$$

D. IBP relations

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&= \int \mathrm{d}^d k \frac{d-2n}{(-k^2+m^2)^n} + \frac{2nm^2}{(-k^2+m^2)^{(n+1)}} \\
&= (d-2n)F(n) + 2nm^2 F(n+1)
\end{aligned}
$$

➪ $$F(n) = -\frac{d-2n+2}{(2n-2)m^2} F(n-1)$$

➪ Any $F(n), n > 1$ can be expressed by $F(1)$
$F(n) = 0$ if $n \leq 0$
$F(1) =$ master integral

D. IBP relations

Example: $F(n) = \int \mathrm{d}^d k \frac{1}{(-k^2+m^2)^n}$

$$F(n) \;=\; -\frac{d-2n+2}{(2n-2)m^2} F(n-1)$$

Raising and lowering indices:

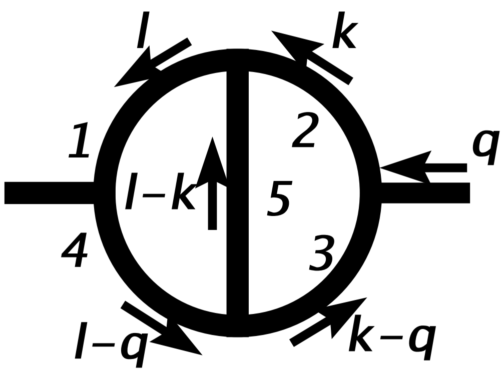
Define operators $\mathbf{1}^{\pm}, \mathbf{2}^{\pm}, \ldots$:

$\mathbf{I}^{\pm} F(n_1,\ldots,n_I,\ldots) = F(n_1,\ldots,n_I \pm 1,\ldots)$

⇨

$$F(n) \;=\; -\frac{d-2n+2}{(2n-2)m^2} \mathbf{1}^- F(n)$$

E. Topology T_1 (massless)



$$== \int \mathrm{d}^d l \int \mathrm{d}^d k \frac{1}{[l^2]^{n_1}[k^2]^{n_2}[(k-q)^2]^{n_3}} \times \frac{1}{[(l-q)^2]^{n_4}[(l-k)^2]^{n_5}}$$

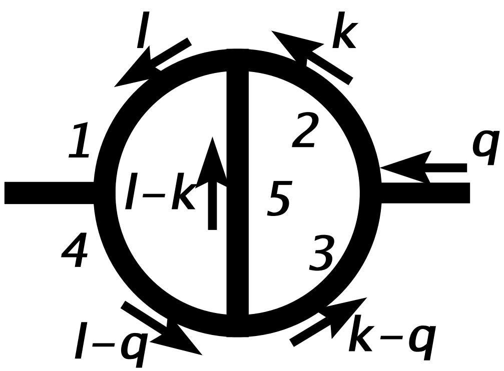
act on integrand with

$$\frac{\partial}{\partial l_\mu}\cdot l_\mu\,, \quad \frac{\partial}{\partial l_\mu}\cdot k_\mu\,, \quad \frac{\partial}{\partial l_\mu}\cdot q_\mu\,, \quad \frac{\partial}{\partial k_\mu}\cdot k_\mu\,, \quad \frac{\partial}{\partial k_\mu}\cdot l_\mu\,, \quad \frac{\partial}{\partial k_\mu}\cdot q_\mu\,,$$

example: $$\frac{\partial}{\partial l_\mu}\cdot l_\mu = d + l_\mu \cdot \frac{\partial}{\partial l_\mu}$$

$$l_\mu \cdot \frac{\partial}{\partial l_\mu}\frac{1}{[l^2]^{n_1}} = -2n_1\frac{1}{[l^2]^{n_1}}$$

Topology T_1 (massless)



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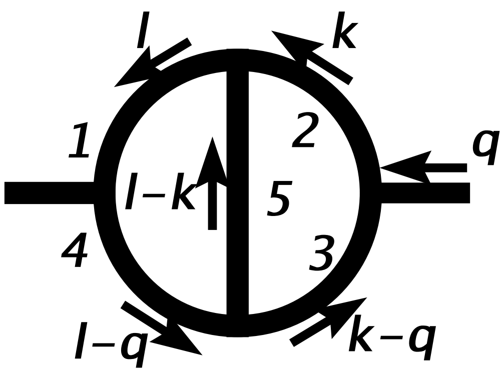
$$l_\mu \cdot \frac{\partial}{\partial l_\mu} \frac{1}{[l^2]^{n_1}} = -2n_1 \frac{1}{[l^2]^{n_1}}$$

$$l_\mu \cdot \frac{\partial}{\partial l_\mu} \frac{1}{[(l-q)^2]^{n_4}} = -2n_4 \frac{1}{[(l-q)^2]^{n_4+1}} \left\{ l^2 - l \cdot q \right\}$$

Topology T_1 (massless)



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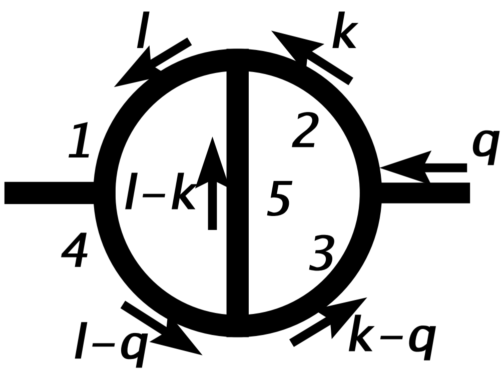
$$l_\mu \cdot \frac{\partial}{\partial l_\mu} \frac{1}{[(l-q)^2]^{n_4}} = -2n_4 \frac{1}{[(l-q)^2]^{n_4+1}} \left\{ l^2 + \frac{1}{2}\left[(l-q)^2 - l^2 - q^2\right] \right\}$$

Topology T_1 (massless)



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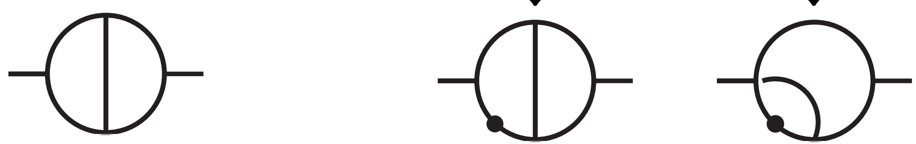
$$l_\mu \cdot \frac{\partial}{\partial l_\mu} \frac{1}{[l^2]^{n_1}} = -2n_1 \frac{1}{[l^2]^{n_1}}$$

$$l_\mu \cdot \frac{\partial}{\partial l_\mu} \frac{1}{[(l-q)^2]^{n_4}} = -2n_4 \underbrace{\frac{1}{[(l-q)^2]^{n_4+1}}}_{\mathbf{4}^+} \left\{ \underbrace{l^2}_{\mathbf{1}^-} + \frac{1}{2}\left[\underbrace{(l-q)^2}_{\mathbf{4}^-} - \underbrace{l^2}_{\mathbf{1}^-} - q^2 \right] \right\}$$

$$\Rightarrow \quad \left[d - 2n_1 - n_4 - n_5 + n_4(q^2 - \mathbf{1}^-)\mathbf{4}^+ + n_5(\mathbf{2}^- - \mathbf{1}^-)\mathbf{5}^+ \right] T_1 = 0$$

IBP identities for T_1

$$\left[\underbrace{d - 2n_1 - n_4 - n_5}_{} + \underbrace{n_4(q^2}_{} - \underbrace{\mathbf{1}^-)\mathbf{4}^+}_{} + n_5(\mathbf{2}^- - \mathbf{1}^-)\mathbf{5}^+ \right] T_1 = 0$$



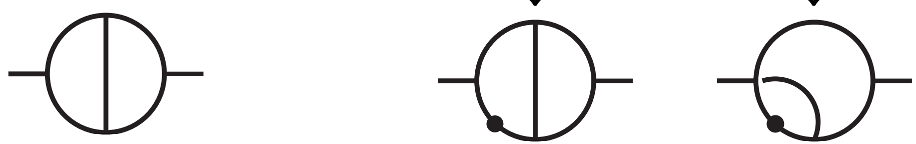
task: combine all IBP identities such that

$T_1(n_1, n_2, n_3, n_4, n_5) \to$ "simpler" integrals, i.e.

- convolutions of 1-loop
- low values of n_i

IBP identities for T_1

$$\left[\underbrace{d-2n_1-n_4-n_5}+\underbrace{n_4(q^2}-\underbrace{\mathbf{1}^-)\mathbf{4}^+}+n_5(\mathbf{2}^--\mathbf{1}^-)\mathbf{5}^+\right]T_1=0$$



task: combine all IBP identities such that

$T_1(n_1,n_2,n_3,n_4,n_5) \to$ "simpler" integrals, i.e.

- convolutions of 1-loop
- low values of n_i

Exercise: compute IBP with "$\frac{\partial}{\partial l_\mu} k_\mu$"

IBP identities for T_1

$$d - 2n_1 + n_4\left(-1 + \left(\boxed{q^2} - \mathbf{1}^-\right)\boxed{\mathbf{4}^+}\right) + n_5\left(-1 + \left(-\mathbf{1}^- + \mathbf{2}^-\right)\mathbf{5}^+\right) = 0$$

$$n_1\left(-1 + \left(-q^2 + \mathbf{4}^-\right)\mathbf{1}^+\right) + n_4\left(1 + \left(q^2 - \mathbf{1}^-\right)\mathbf{4}^+\right) + \left(-\mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^- + \mathbf{4}^-\right)n_5\,\mathbf{5}^+ = 0$$

$$n_1\left(-1 + \left(-\mathbf{2}^- + \mathbf{5}^-\right)\mathbf{1}^+\right) + \left(\boxed{q^2} - \mathbf{1}^- - \mathbf{3}^- + \mathbf{5}^-\right)n_4\,\boxed{\mathbf{4}^+} + n_5\left(1 + \left(-\mathbf{1}^- + \mathbf{2}^-\right)\mathbf{5}^+\right) = 0$$

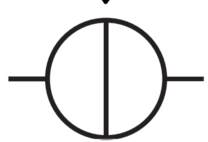
$$n_2\left(-1 + \left(-\mathbf{1}^- + \mathbf{5}^-\right)\mathbf{2}^+\right) + \left(q^2 - \mathbf{2}^- - \mathbf{4}^- + \mathbf{5}^-\right)n_3\,\mathbf{3}^+ + n_5\left(1 + \left(\mathbf{1}^- - \mathbf{2}^-\right)\mathbf{5}^+\right) = 0$$

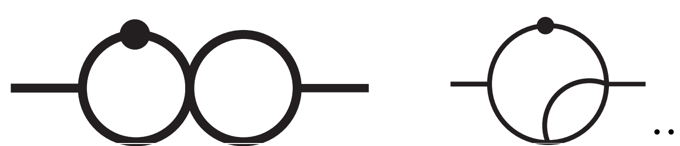
$$n_2\left(-1 + \left(-q^2 + \mathbf{3}^-\right)\mathbf{2}^+\right) + n_3\left(1 + \left(q^2 - \mathbf{2}^-\right)\mathbf{3}^+\right) + \left(\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^- - \mathbf{4}^-\right)n_5\,\mathbf{5}^+ = 0$$

$$d - 2n_2 + n_3\left(-1 + \left(q^2 - \mathbf{2}^-\right)\mathbf{3}^+\right) + n_5\left(-1 + \left(\mathbf{1}^- - \mathbf{2}^-\right)\mathbf{5}^+\right) = 0$$

(1) – (3):

$$\Big[\underbrace{d - 2n_5 - n_1 - n_4} - \underbrace{n_1(\mathbf{5}^- - \mathbf{2}^-)\mathbf{1}^+ - n_4(\mathbf{5}^- - \mathbf{3}^-)\mathbf{4}^+}\Big]T_1 = 0$$





F. Triangle rule

$$T(n_1,n_2,n_3,n_4,n_5) = \frac{1}{d-2n_5-n_1-n_4}$$

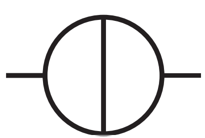
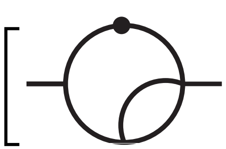
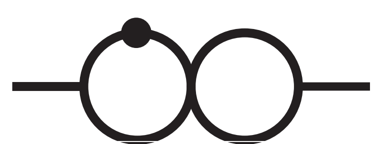
$$\times \left[n_1(\mathbf{5}^- - \mathbf{2}^-)\mathbf{1}^+ + n_4(\mathbf{5}^- - \mathbf{3}^-)\mathbf{4}^+ \right] T_1(n_1,n_2,n_3,n_4,n_5)$$

$\rightarrow$ recurrence relation

example: $n_1 = n_2 = n_3 = n_4 = n_5 = 1$

$$T_1(1,1,1,1,1) = \frac{1}{d-4}\Big[T_1(2,1,1,1,0) - T_1(2,0,1,1,1)$$

$$+ T_1(1,1,1,2,0) - T_1(1,1,0,2,1)\Big]$$

 $= \frac{1}{\epsilon}\Big[$  $-$  $\Big] \sim 6\zeta(3)$

Triangle rule

indices > 1 $\rightarrow$ apply rec.-rel. repeatedly

$$
\begin{array}{lllll}
T_1(1,3,1,1,1) & \rightarrow & T_1(2,3,1,1,0) & & \\
 & & T_1(2,2,1,1,1) & \rightarrow & T_1(3,2,1,1,0) \\
 & & T_1(1,3,1,2,0) & & T_1(3,1,1,1,1) \rightarrow T_1(4,1,1,1,0) \\
 & & T_1(1,3,0,2,1) & & T_1(2,2,1,2,0) \qquad\quad\; T_1(4,0,1,1,1) \\
 & & & & T_1(2,2,0,2,1) \qquad\quad\; T_1(3,1,1,2,0) \\
 & & & & \qquad\qquad\qquad\qquad\quad\; T_1(3,1,0,2,1)
\end{array}
$$

$\rightarrow$ generates many terms

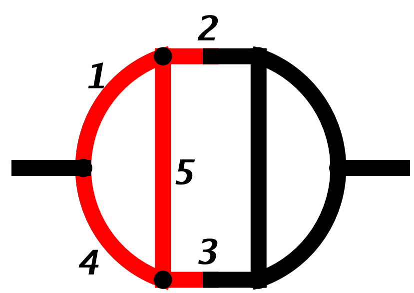
$\rightarrow$ needs computer algebra!

$$\left[n_1(\mathbf{5}^- - \mathbf{2}^-)\mathbf{1}^+ + n_4(\mathbf{5}^- - \mathbf{3}^-)\mathbf{4}^+ \right]$$

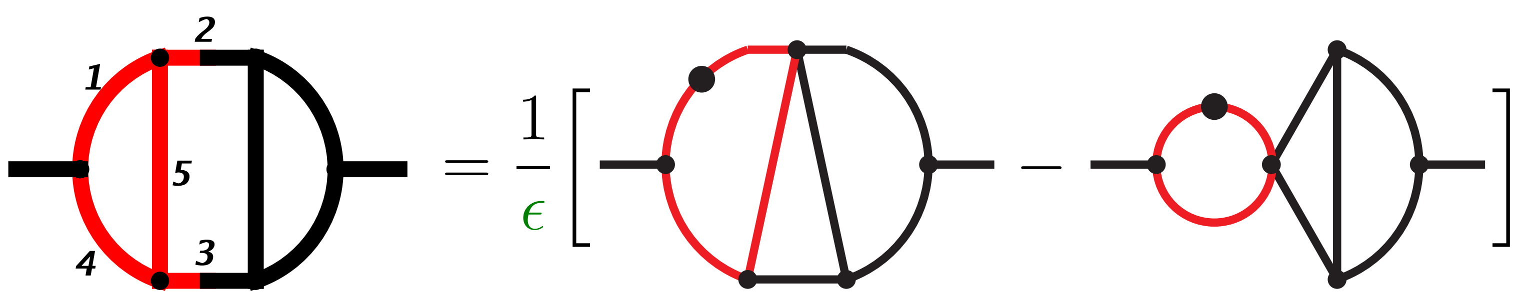
Triangle rule

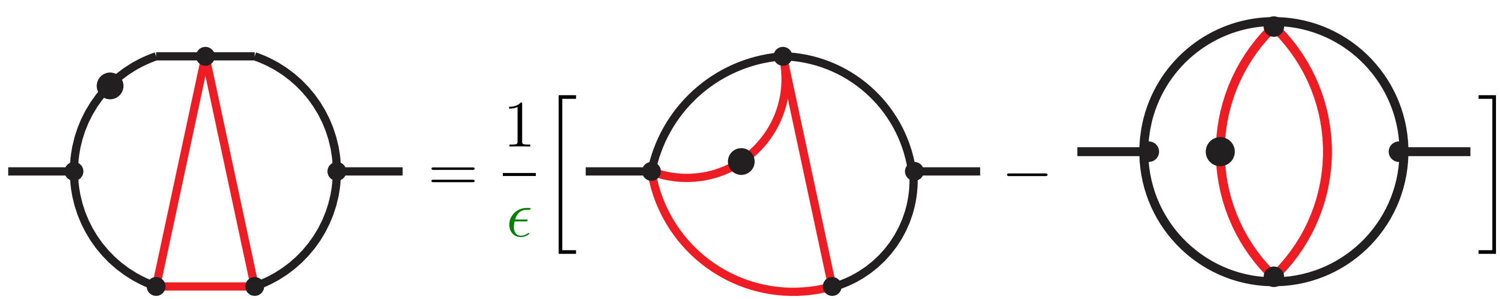
observation: this rec.-relation applies to any triangular sub-loop!

e.g. 3-loop:



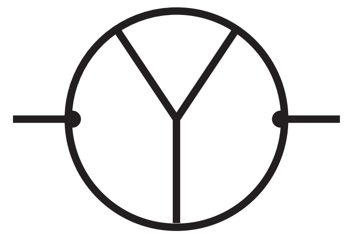
$$d - 2n_5 - n_1 - n_4 = n_1(\mathbf{5}^- - \mathbf{2}^-)\mathbf{1}^+ + n_4(\mathbf{5}^- - \mathbf{3}^-)\mathbf{4}^+$$





Triangle rule – exercise

Which convolutions of 1- and 2-loop integrals does the following integral reduce to?



G. Reduction to MIs

- “modern”: Laporta-, Gröbner-, Baikov-, . . . algorithm
- “old-fashioned” (but much more effective):
  look (“by hand”) for combinations of IBP relations which systematically reduce complicated integrals to simpler ones

`MINCER`: massless 2-point function; 1, 2, 3 loops

[Larin,Tkachov,Vermaseren]

`Grinder`: HQET integrals; 1, 2, 3 loops [Grozin]

`MATAD`: vaccum bubbles; 1, 2, 3 loops [Steinhauser]

II. MATAD

(MAssive TADpoles)

[MATAD3 for FORM3]

MATAD: MAssive TADpoles

- written in FORM (MATAD3 $\leftrightarrow$ FORM3)
- vaccum bubbles up to 3 loops
- propagators: mass M1 or massless
- zero external momentum
- final result up to (incl.) $\mathcal{O}(\epsilon^{3-l})$

MATAD: applications

- moments of $\Pi_\gamma(q^2)$ ➪ m_c, m_b
- ρ parameter: 3-loop corrections
- decoupling of α_s and m_q
- matching coefficient of eff. Lagrangian ($gg \to H$, $\bar{B} \to X_s\gamma$)
- $\Gamma(H \to \gamma\gamma)$
- ...

A. Flowchart

read FD
translate notation
traces
expansion
Wick rotation
. . .

← FD

← projectors
← FRs
← . . .

↓

reduction
MIs
simple integrals

↓

expansion in ϵ
print and save
result

Input files:
– `.dia` file: 1 `FORM` fold/diagram
– `GLOBAL`: overall settings, for interaction, projectors, . . .
“`TREAT`”-files
– `declare` file for user-defined variables

A. Flowchart

read FD
translate notation
traces

← FD

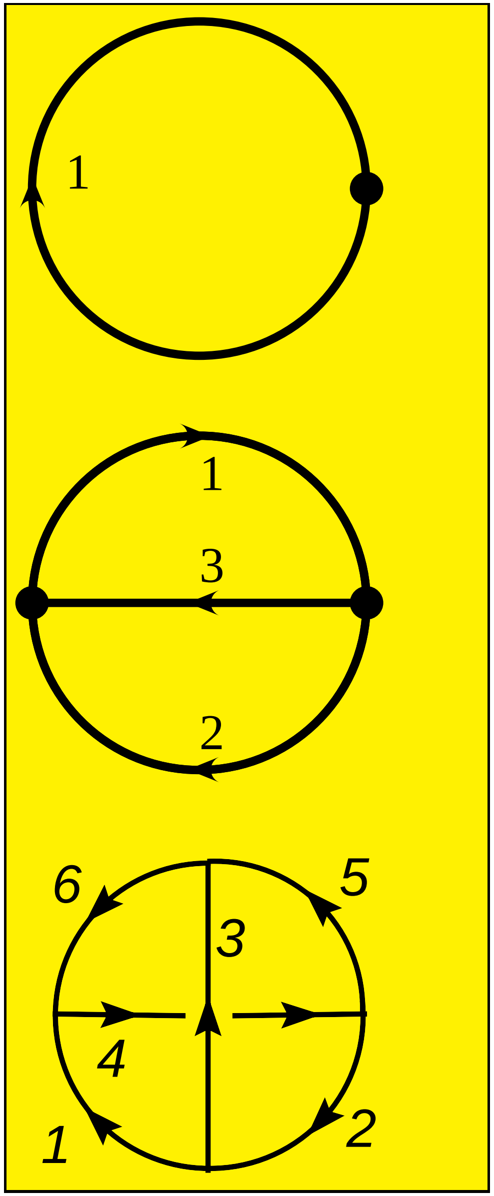
← projectors

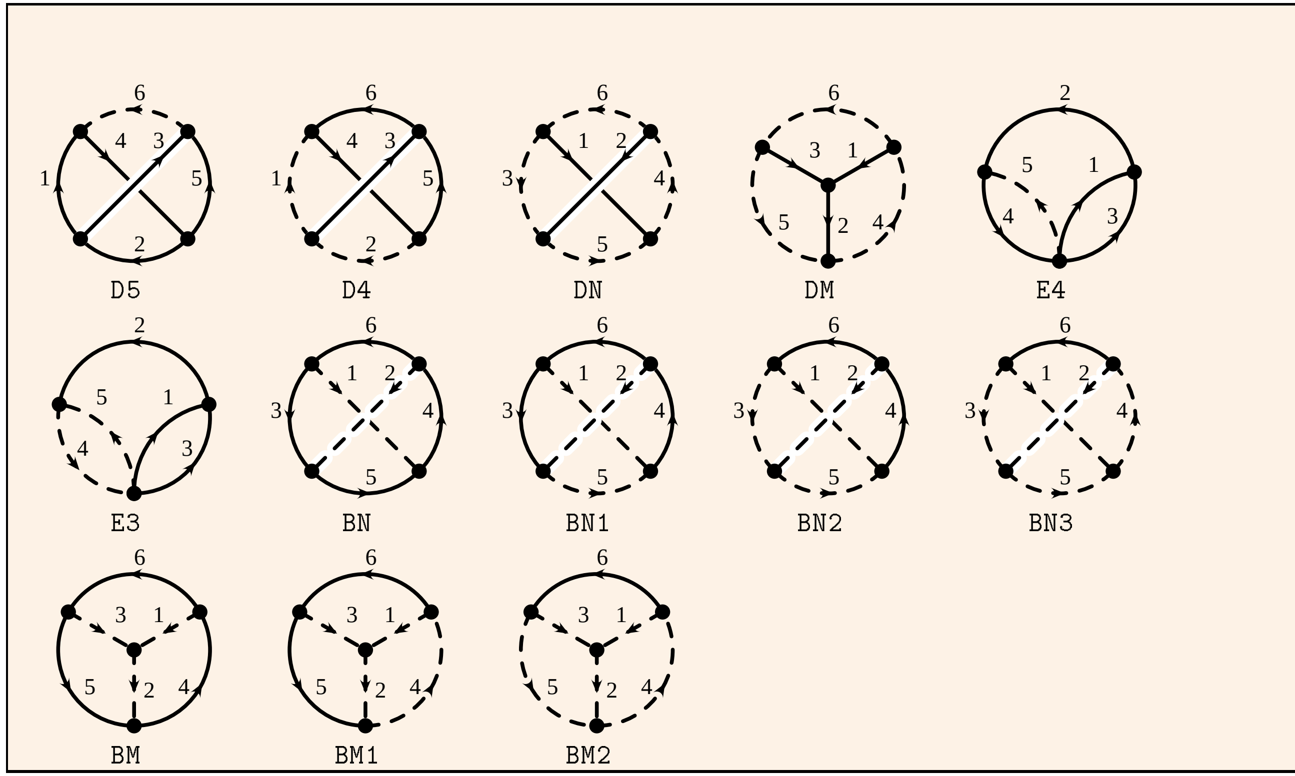
```
*--#[ GLOBAL :
  #define LOWLIMM1 "-2"
  #define GAUGE "xi"
*--#] GLOBAL :
*--#[ TREAT1 :
  multiply, deno(3,-2)*( -d_(mu1,mu2)
    + q1(mu1)*q1(mu2)/q1.q1 );
  .sort
*--#] TREAT1 :
```

expansion in ϵ
print and save
result

...am
"TREAT"-files
– declare file for user-defined variables

B. Notation/Topologies

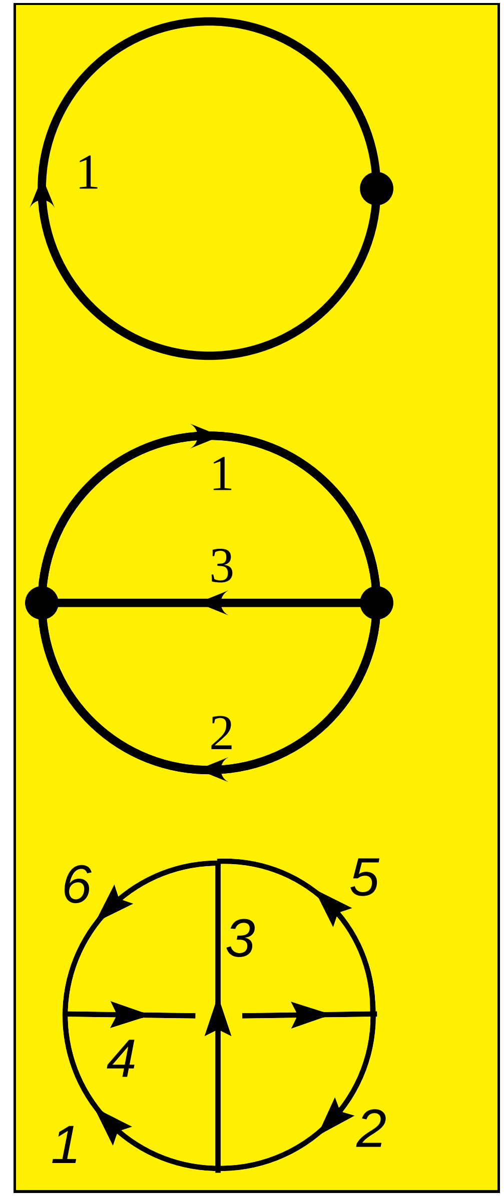




B. Notation

- masses: `M1,M2,...`
- `eM1` $= \left(\frac{\mu^2}{M_1^2}\right)^\epsilon$, `...`
- loop momenta: `p11,p12,...`
- external momenta: `q1,q2,...`
- massless scalar denominators: `1/p11.p11,1/p12.p12,...`
- massive scalar denominators: `s11m1,s12m1,...`; `s12m1` $\equiv \frac{1}{M_1^2 - p_2^2}$
- internal indices: `nu1,nu2,...`
- external indices: `mu1,mu2,...`
- `deno(a,b)` $= \frac{1}{a+b\epsilon}$
- scalar denominator:

  Den(L,#,⟨momenta,masses⟩,exp,⟨momenta,masses⟩),#,L)



B. Notation

- scalar denominator:

  `Den(L,#,`$\langle$`momenta,masses`$\rangle$`,exp,`$\langle$`momenta,masses`$\rangle$`),#,L)`

  example:

  `Den(L,28,+p15,pM1,28,L)` $\equiv \frac{1}{-p_5^2}$

  `Den(L,28,-p17,pM1,M1,pM1,28,L)` $\equiv \frac{1}{M_1^2-p_7^2}$

  `Den(L,32,-p16,pM1,M1,pM1,exp,+q1,pQ1,32,L)`

  $\equiv \frac{1}{M_1^2-(-p_6+q_1)^2}$ expand in q_1

  `pM1,pQ1`: “powercount” variables; needed for expansion

- spinor lines: `FT1, FT2, ...`

  `FT1(L,32,+p15,pM1,M1,pM1,32,L)` $\equiv \frac{1}{M_1-\gamma_\nu p_5^\nu}$

  `FT1(L,26,-p13,pM1,M1,pM1,exp,+q1,pQ1,26,L)`

  $\equiv \frac{1}{M_1-\gamma_\nu(-p_3+q_1)^\nu}$

- `FT1(mu1)` $\equiv \gamma^{\mu_1}$

B. Notation

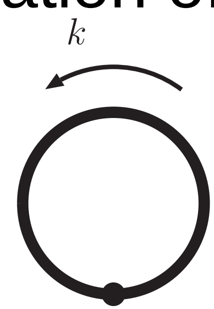
- gluon propagator:
  `Dg(nu13,nu14,L,26,+p12,pM1,26,L)`
  $\equiv \frac{-g^{\nu 13 \nu 14} - \text{'GAUGE'} p_2^{\nu 13} p_2^{\nu 14}}{-p_2^2}$
- `Vgh`: ghost-gluon vertex
- `V3g`: 3-gluon vertex
- `Vggs, Dsig`: implementation of 4-gluon vertex
- other vertices: use “`TREAT0`”

B. Notation

- colour diagram: `fqcd...`
  - fundamental, external: `i1,i2,...`
  - fundamental, internal: `j1,j2,...`
  - adjoint, external: `a(1),a(2),...`
  - adjoint, internal: `b(1),b(2),...`
  - `d_(i1,i2)` $\equiv \delta^{i_1,i_2}$
  - `GM(a(1),i1,i2)` $\equiv \lambda^{a_1}_{i_1,i_2}$
  - `prop(a(1),a(2)))` $\equiv \delta^{a_1,a_2}$

More notation/convention

- imaginary unit i:
  - $\frac{1}{M^2-p^2} \quad \stackrel{W.R.}{\longrightarrow} \quad \frac{1}{M^2+p^2}$
  - all vertices: $i \cdot (\ldots)$
    all propagators: $\frac{1}{i} \cdot (\ldots)$
    ➪ don't care about "i" of internal lines
  - note: $i \left(\frac{1}{i} \int \mathrm{d}^d k_j\right)^{l=1,2,3} \quad \stackrel{W.R.}{\longrightarrow} \quad i \left(\int \mathrm{d}^d k_{j,E}\right)^{l=1,2,3}$
    overall i (independent of l) is omitted
- γ_E
  multiply with $(e^{\gamma_E \epsilon})^l$ ➪ γ_E disappears from final result
- normalization of the integrals: $\int \frac{\mathrm{d}k}{i\pi^{d/2}}$
  e.g.:

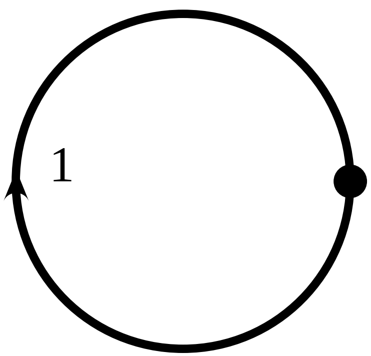


$$= e^{\gamma_E \epsilon}\, \mathtt{eM1}\, (\mathtt{M1}^2)^{(2-n)} \frac{\Gamma(n-d/2)}{\Gamma(n)}$$

```
http://www-ttp.particle.uni-karlsruhe.de/˜ms/software.html

1. download matad3.tar.gz

2. mkdir matad3; cd matad3

3. gunzip matad3.tar.gz

4. tar -xvf matad3.tar

5. adapt path to ’form3’ in startform3
```

C. Examples

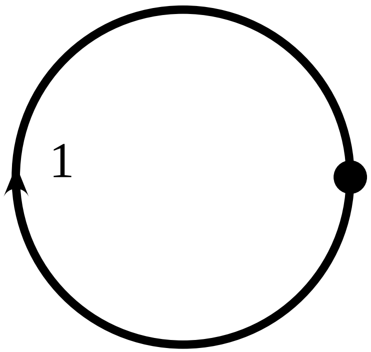
1-loop tadpole




```
*--#[ d1l1 :
s11m1
;
#define INT1 "tad1l"
#define MASS1 "M1"
*--#] d1l1 :

*--#[ expd1l1 :
multiply, eM1*pM1^4;
#call denoexp{M1}
#include matad.info # time
*--#] expd1l1 :
```

C. Examples

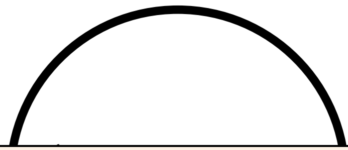
1-loop tadpole




```
> cd calc3

> startform3 -S form.set -d CLASS=c_misc -d PROBLEM=test
-d LOOPS=1 -d DIAFILE=test.dia
-d RESDIR=results -d DIAGRAM=d1l1 generic/maindia.frm
```

C. Examples

1-loop tadpole

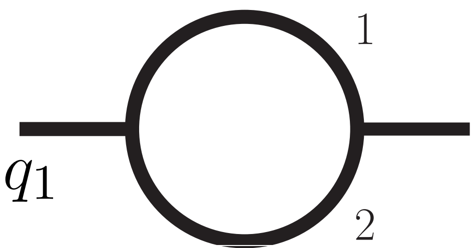


```
d1l1 =
+ ep^-1 * ( - M1^2*eM1 )

+ ep * ( - 1/2*M1^2*eM1*z2 - M1^2*eM1 )

+ ep^2 * ( - 1/2*M1^2*eM1*z2 + 1/3*M1^2*eM1*z3 - M1^2*eM1 )

- M1^2*eM1;
```

```
-d RESDIR=results -d DIAGRAM=d1l1 generic/maindia.frm
```

C. Examples (2)

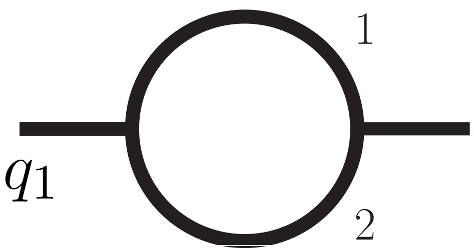
1-loop expanded in external momentum




```
*--#[ d1l2 :

Den(L,11,p11,pM1,M1,pM1,11,L) *
Den(L,22,p11,pM1,M1,pM1,exp,q1,pQ1,22,L)
;

#define INT1 "tad1l"
#define MASS1 "M1"

*--#] d1l2 :
```

C. Examples (2)

1-loop expanded in external momentum

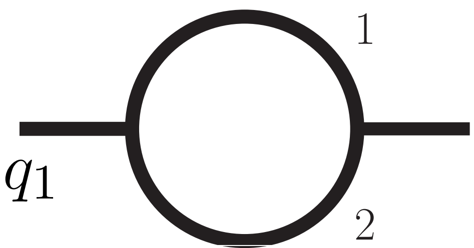



```
> cd calc3
```

```
> startform3 -S form.set -d CLASS=c_misc -d PROBLEM=test
-d LOOPS=1 -d DIAFILE=test.dia
-d RESDIR=results -d DIAGRAM=d1l2 generic/maindia.frm
```

C. Examples (2)

1-loop expanded in external momentum



Q1: Euclid. momentum

```
d1l2 =
+ ep^-1 * ( eM1 )

+ ep * ( 1/2*eM1*z2 + 1/60*Q1.Q1^2*M1^-4*eM1 )

+ ep^2 * ( - 1/3*eM1*z3 - 1/12*Q1.Q1*M1^-2*eM1*z2 + 1/120*Q1.Q1^2*
M1^-4*eM1*z2 )

- 1/6*Q1.Q1*M1^-2*eM1 + 1/60*Q1.Q1^2*M1^-4*eM1;
```

D. (MIs) Symbols in MATAD

$$\texttt{D6} = 6\zeta_3 - 17\zeta_4 - 4\zeta_2 \ln^2 2 + \frac{2}{3}\ln^4 2 + 16\mathrm{Li}_4\left(\frac{1}{2}\right) - 4\left[\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right]^2$$

$$\texttt{D5} = 6\zeta_3 - \frac{469}{27}\zeta_4 + \frac{8}{3}\left[\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right]^2 - 16\sum_{m>n>0}\frac{(-1)^m\cos(2\pi n/3)}{m^3 n} \approx -8.2168598175087380629133983386 01$$

$$\texttt{D4} = 6\zeta_3 - \frac{77}{12}\zeta_4 - 6\left[\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right]^2$$

$$\texttt{D3} = 6\zeta_3 - \frac{15}{4}\zeta_4 - 6\left[\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right]^2$$

$$\texttt{DM} = 6\zeta_3 - \frac{11}{2}\zeta_4 - 4\left[\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right]^2$$

$$\texttt{DN} = 6\zeta_3 - 4\zeta_2 \ln^2 2 + \frac{2}{3}\ln^4 2 - \frac{21}{2}\zeta_4 + 16\mathrm{Li}_4\left(\frac{1}{2}\right)$$

$$\texttt{B4} = -4\zeta_2 \ln^2 2 + \frac{2}{3}\ln^4 2 - \frac{13}{2}\zeta_4 + 16\mathrm{Li}_4\left(\frac{1}{2}\right)$$

$$\texttt{E3} = -\frac{139}{3} - \frac{\pi\sqrt{3}\ln^2 3}{8} - \frac{17\pi^3\sqrt{3}}{72} - \frac{21}{2}\zeta_2 + \frac{1}{3}\zeta_3 + 10\sqrt{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - 6\sqrt{3}\mathrm{Im}\left[\mathrm{Li}_3\left(\frac{e^{-i\pi/6}}{\sqrt{3}}\right)\right]$$

$$\texttt{S2} = \frac{4}{9\sqrt{3}}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)$$

$$\texttt{OepS2} = -\frac{763}{32} - \frac{9\pi\sqrt{3}\ln^2 3}{16} - \frac{35\pi^3\sqrt{3}}{48} + \frac{195}{16}\zeta_2 - \frac{15}{4}\zeta_3 + \frac{57}{16}\zeta_4 + \frac{45\sqrt{3}}{2}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - 27\sqrt{3}\mathrm{Im}\left[\mathrm{Li}_3\left(\frac{e^{-i\pi/6}}{\sqrt{3}}\right)\right]$$

$$\texttt{T1ep} = -\frac{45}{2} - \frac{\pi\sqrt{3}\ln^2 3}{8} - \frac{35\pi^3\sqrt{3}}{216} - \frac{9}{2}\zeta_2 + \zeta_3 + 6\sqrt{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - 6\sqrt{3}\mathrm{Im}\left[\mathrm{Li}_3\left(\frac{e^{-i\pi/6}}{\sqrt{3}}\right)\right]$$

Literature

- M. Steinhauser, “MATAD: A program package for the computation of massive tadpoles,” Comput. Phys. Commun. **134** (2001) 335 [arXiv:hep-ph/0009029].
- `http://www-ttp.particle.uni-karlsruhe.de/~ms/software.html`
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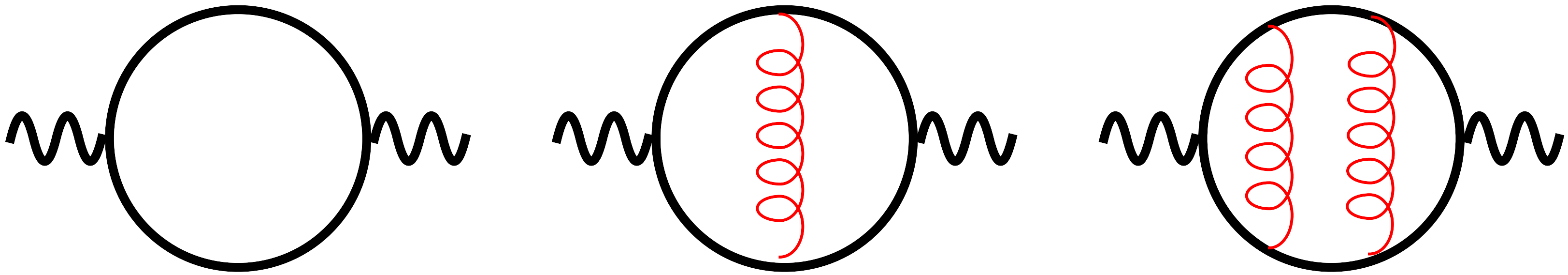
- S. A. Larin, F. V. Tkachov and J. A. M. Vermaseren, “The Form Version Of Mincer,”
- S. G. Gorishnii, S. A. Larin, L. R. Surguladze and F. V. Tkachov, “MINCER: Program for Multiloop Calculations in Quantum Field Theory for the Schoonschip System,” Comput. Phys. Commun. **55** (1989) 381.
- K. G. Chetyrkin and F. V. Tkachov, “Integration By Parts: The Algorithm To Calculate Beta Functions In 4 Loops,” Nucl. Phys. B **192** (1981) 159.
- D. J. Broadhurst, “Three Loop On-Shell Charge Renormalization Without Integration: Lambda-Ms (QED) To Four Loops,” Z. Phys. C **54**, 599 (1992).
- L. V. Avdeev, “Recurrence Relations for Three-Loop Prototypes of Bubble Diagrams with a Mass,” Comput. Phys. Commun. **98** (1996) 15 [arXiv:hep-ph/9512442].

III. Examples

A. Scalar 3-loop integral

see: `problems/c_misc/test`

B. Photon polarization fucntion



$$
\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right)\,\Pi(q^2) = i\int \mathrm{d}x\, e^{iqx}\langle 0|T j_\mu(x) j_\nu^\dagger(0)|0\rangle
$$
$$
= \Pi^{(0)}(q^2) + \frac{\alpha_s}{4\pi}\Pi^{(1)}(q^2) + \left(\frac{\alpha_s}{4\pi}\right)^2 \Pi^{(2)}(q^2)
$$
$$
= +\ldots
$$

Projector:

$$
q^2\Pi(q^2) = \frac{1}{d-1}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\Pi^{\mu\nu}(q^2)
$$

see: `problems/c_pi/pi`

Exercises I

1. Compute the ϵ expansion for the one-loop tadpole with index $n = 1, 2, 3, 4, \ldots$ using
   (a) the explicit analytic result,
   (b) the recurrence relation.
2. Derive the triangle rule for the two-loop integral of type T1 where each line has a different mass.
3. (a) Derive the recurrence relations for the massless two-loop integral of type T1.
   (b) Program the triangle rule in order to compute the ϵ expansion of the integral T1 for arbitrary indices up to (including) terms of order ϵ^2.
4. (a) Derive the recurrence relations for the two-loop tadpole with 2 massive and 1 massless lines.
   (b) Compare with explicit analytic result.

Exercises II

1. Compute
   (a) one-loop tadpole,
   (b) two-loop tadpole with 2 massive and 1 massless line
   (c) three-loop tadpole with 4 lines where each of them connects vertex 1 with vertex 2; 2 massive and 2 massless lines,

   with `MATAD` for various choices of the indices. Compare with the analytic results.
2. Compute three-loop vacuum integrals (with six lines, "`tad3l`") with various massive/massless combinations. Which constant appears in the final result? For which choice of indices is the result finite?
3. Check that the sum of the (two) singlet diagrams contributing to $\Pi(q^2)$ at three loops is zero. Why?
4. Compute $\Pi(q^2)$ for arbitrary gauge parameter ξ and check that ξ drops out in the sum of the bare diagrams.
5. Renormalize $\Pi(q^2)$ and compare with the result in the literature.
6. Compute the axial-vector, scalar and pseudo-scalar correlator.